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Polyhedral graphs with restricted number of faces of the same type

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Abstract

Let $G = G(V,E,F)$ be a polyhedral graph with vertex set V, edge set E and face set F. A face α is an $\langle a_1,\ldots,a_l\rangle$-*face* if α is an l-gon and the degrees $d(x_i)$ of the vertices $x_1,\ldots,x_l$ incident with α in the cyclic order are $a_1,\ldots,a_l$, respectively. The lexicographic minimum $\langle b_1,\ldots,b_l\rangle$ such that α is a $\langle b_1,\ldots,b_l\rangle$-face is called the *type* of α. Furthermore let z be a given integer. We consider polyhedral graphs where the number of faces of each type is restricted by z. We prove that there is only a finite number of such graphs.

1. Introduction

All graphs considered in the sequel are polyhedral graphs, i.e. they are planar and three-connected. Vertex set, edge set and face set are denoted by V, E and F, respectively. For a face $\alpha \in F$, $d(\alpha)$ is the number of vertices incident with α and for a vertex $x \in V$, $d(x)$ is the degree of x. If $x \in V$ is incident with α we write $x \in \alpha$.

A graph $G(V,E,F)$ is called *face transitive* if for each pair of faces, $\alpha,\beta \in F$ there is an automorphism $\phi_{\alpha,\beta}: V \to V$ such that $\phi_{\alpha,\beta}(\alpha) = \beta$.

A face α is an $\langle a_1,\ldots,a_l\rangle$-*face* if α is an l-gon and the degrees $d(x_i)$ of the vertices $x_1,\ldots,x_l$ incident with α in the cyclic order are $a_1,\ldots,a_l$, respectively. Obviously, α is also an $\langle a_2,a_3,\ldots,a_l,a_1\rangle$-face, an $\langle a_3,a_4,\ldots,a_l,a_1,a_2\rangle$-face,…, and an $\langle a_l,a_{l-1},\ldots,a_2,a_1\rangle$-face. The lexicographic minimum $\langle b_1,\ldots,b_l\rangle$ such that α is a $\langle b_1,\ldots,b_l\rangle$-face is called the *type* of α.

Let $z_{\langle b_1,\ldots,b_l\rangle}$ be the number of faces of type $\langle b_1,\ldots,b_l\rangle$ in G. A planar 3-connected graph G is called *oblique* if no two faces are of the same type, G is called z-*oblique*, if $z_{\langle b_1,\ldots,b_l\rangle} \leqslant z$ for all types of faces of G.

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Grünbaum and Shephard [2] have described symmetries of polyhedral graphs, covering all face transitive graphs.

In [5] are listed all polyhedral graphs having only faces of the same type. It has been shown that there is exactly one non-face transitive polyhedral graph G with 24 faces having only one type of faces, namely all faces of G are of type $\langle 3,4,4,4\rangle$.

The opposite case when no two faces of G have the same type is considered in [5] too. It was shown, that the set of oblique graphs is not empty. Borodin [1] proved that in a planar graph of minimum degree 5 we always have $18z_{\langle 5,5,5\rangle} + 9z_{\langle 5,5,6\rangle} + 5z_{\langle 5,5,7\rangle} + 4z_{\langle 5,6,6\rangle} \geqslant 144$. Thus, the set of z-oblique graphs for $z \leqslant 3$ having neither vertices of degree 3 nor vertices of degree 4 is empty.

In [5] it is proved that the set of oblique triangulations having no vertex of degree 3 is empty, and that the set of oblique triangulations is finite but not empty. If we consider a set of graphs we mean a set of pairwise non-isomorphic graphs. In this paper we shall prove

Theorem 1. *For every z, the set of z-oblique graphs is finite.*

Let us consider an infinite set $\mathscr{G}$ of pairwise non-isomorphic polyhedral graphs and let k be any integer. Using Theorem 1 it follows that $\mathscr{G}$ contains a graph having at least k faces of the same type. In fact, even infinitely many such graphs must belong to $\mathscr{G}$. Note that there is no analogous result if the connectivity of G is at most 2. For $k \geqslant 3$ let G_k be the graph consisting of two vertices v and w joined by k paths $P_1,\ldots,P_k$ where the number of edges of P_i is i for $i=1,\ldots,k$. Thus G_k is planar, 2-connected and every embedding of G_k in the plane has no two faces of the same type.

2. Proof of the main result

Using a discharging method, we shall prove that for every z and for every z-oblique graph the total number of faces as well as the size $d(\alpha)$ of occurring faces are bounded by constants depending on z.

Let us define charges $w(\alpha)$ and $w(x)$ for faces α and vertices x, respectively, in the following way:

$$d^*(x) := \begin{cases} d(x), & d(x) < 42 + 21z, \\ d(x) - 3z & \text{otherwise}, \end{cases}$$

$$w(x) := \begin{cases} 0 & d(x) < 42 + 21z, \\ 3z & \text{otherwise}, \end{cases}$$

$$w(\alpha) := 2(d(\alpha) - 3) + \sum_{x\in\alpha} \frac{d^*(x) - 6}{d(x)}.$$

Claim 1. $\sum_{\alpha \in F} w(\alpha) + \sum_{x \in V} w(x) = -12$.

Proof. From Euler's formula we know $\sum_{\alpha \in F} 2(d(\alpha) - 3) + \sum_{x \in V}(d(x) - 6) = -12$. Furthermore $d(x) - 6 = d^*(x) - 6 + w(x)$ for every vertex $x \in V$ and $\sum_{x \in V} d^*(x) - 6 = \sum_{\alpha \in F} \sum_{x \in \alpha}(d^*(x) - 6)/d(x)$. $\square$

Denote the set of k-gons of G by F_k and let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_k$ be the degrees of the vertices incident with a face $\alpha \in F_k$. Note that such a degree sequence may represent distinct types of faces, for example $d_1 = 3$, $d_2 = d_3 = 4$, $d_4 = 5$ corresponds to $\langle 3,4,4,5 \rangle$ and $\langle 3,4,5,4 \rangle$.

Using Euler's formula Lebesgue [3,4] gives a list of types of faces such that at least one of these faces has to occur in every plane graph. These faces are exactly the faces of negative charge by our definition. We would like to claim that there is only a finite number of types of such faces. Unfortunately, this has not been true so far. If $\alpha \in F_3$ and $d_1 = 3$, $d_2 \leqslant 6$ or $d_1 = d_2 = 4$ then the charge of α remains negative even for arbitrarily high degree of the third vertex. A similar situation occurs for $\alpha \in F_4$ and $d_1 = d_2 = d_3 = 3$. However, if α contains a vertex of high degree ($\geqslant 42 + 21z$) then this vertex has some charge and we may redistribute this charge to the mentioned types of faces to make the charge of these faces non-negative. Note that at most z faces of the same type have a common vertex of high degree because of the restriction for the number of faces of the same type.

Discharging rules:

1. $d(x) \geqslant 42 + 21z$. Set $w'(x) := w(x)$
   (a) if $x \in \alpha \in F_3$ and $d_1 = d_2 = 3$ then
   $w'(\alpha) := w(\alpha) + 8/7$ and $w'(x) := w'(x) - 8/7$.
   (b) if $x \in \alpha \in F_3$ and $d_1 = 3$, $d_2 = 4$ then
   $w'(\alpha) := w(\alpha) + 5/7$ and $w'(x) := w'(x) - 5/7$.
   (c) if $x \in \alpha \in F_3$ and $d_1 = 3$, $d_2 = 5$ then
   $w'(\alpha) := (\alpha) + 3/7$ and $w'(x) := w'(x) - 3/7$.
   (d) if $x \in \alpha \in F_3$ and $d_1 = 3$, $d_2 = 6$ then
   $w'(\alpha) := w(\alpha) + 1/7$ and $w'(x) := w'(x) - 1/7$.
   (e) if $x \in \alpha \in F_3$ and $d_1 = d_2 = 4$ then
   $w'(\alpha) := w(\alpha) + 1/7$ and $w'(x) := w'(x) - 1/7$.
   (f) if $x \in \alpha \in F_4$ and $d_1 = d_2 = d_3 = 3$ then
   $w'(\alpha) := w(\alpha) + 1/7$ and $w'(x) := w'(x) - 1/7$.
2. $w'(x) = w(x)$ if $d(x) < 42 + 21z$ and
   $w'(\alpha) = w(\alpha)$ for all faces not mentioned above.

Clearly, the sum over all charges has not changed.

Claim 2. $\sum_{\alpha \in F} w'(\alpha) + \sum_{x \in V} w'(x) = \sum_{\alpha \in F} w(\alpha) + \sum_{x \in V} w(x) = -12$.

Furthermore, the charge of all vertices is still non-negative.

Table 1
Degree sequences of 3-faces with negative charge and minimum degree 3

$d_1 =$	3	3	3	3	3	3	3	3	3
$d_2 =$	3	4	5	6	7	8	9	10	11
$d_3 <$	$42+21z$	$42+21z$	$42+21z$	$42+21z$	42	24	18	15	14

Table 2
Degree sequences of 3-faces with negative charge and minimum degree 4 or 5

$d_1 =$	4	4	4	4	5	5
$d_2 =$	4	5	6	7	5	6
$d_3 <$	$42+21z$	20	12	10	10	8

Claim 3.

$$w'(x)\begin{cases} =0, & d(x) < 42+21z,\\ \geqslant (2/7)z, & d(x) \geqslant 42+21z. \end{cases}$$

Proof. If $d(x) < 42+21z$ then the charge is obviously 0. If $d(x) \geqslant 42+21z$ then $w(x)=3z$ and we have to consider the discharging rules. Note that a vertex x is incident with at most z faces of each type and that each of the rules 1(a)–1(f) concerns only one type of faces. Thus $w'(x) \geqslant 3z-(8/7)z-(5/7)z-(3/7)z-3*(1/7)z=(2/7)z$. $\square$

Now we consider the new charge of faces. First, let us give an inequality which is important for faces with at least 6 incident vertices.

Claim 4. $w'(\alpha)=w(\alpha) \geqslant d(\alpha)-6$.

Proof. Note that $(d^*(x)-6)/d(x) \geqslant -1$ for every vertex $x \in V$. Thus, $w(\alpha) \geqslant 2(d(\alpha)-3)-d(\alpha)=d(\alpha)-6$. $\square$

Thus, only k-gons with $k \leqslant 5$ may have negative charge $w'(\alpha)$.

In the following tables we give the degree sequences of all faces of negative charge after discharging. The faces with vertex of maximum degree at most 42 correspond to the faces described by Lesbesgue [3,4]. Furthermore our discharging allows to have only a finite number of degree sequences of faces with negative charge because faces with a vertex of degree at least $42+21z$ have always non-negative new charge.

Let F^- be the set of all faces having a degree sequence described in Tables 1–3. Then by the result of Lebesgue [3,4] combined with our discharging we obtain

Lemma 2. *If* $\alpha \in F \setminus F^-$ *then* $w'(\alpha) \geqslant 0$.

So we know that only faces of F^- may have negative charge $w'(\alpha)$. However, the faces of F^- represent only a bounded number of degree sequences, and thus, a bounded number of types of faces. Clearly, this number depends on z. Furthermore,

Table 3
Degree sequences of 5- and 4-faces with negative charge

$d_1 =$	3
$d_2 =$	3
$d_3 =$	3
$d_4 =$	3
$d_5 <$	6

$d_1 =$	3	3	3	3
$d_2 =$	3	3	3	3
$d_3 =$	3	4	5	4
$d_4 <$	$42 + 21z$	12	8	6

there are at most z faces of each type belonging to G. Thus the number of faces α of negative charge $w'(\alpha)$ is bounded by a constant $c_1(z)$ and so the sum of the charges of all these faces is bounded from below by a constant $c_2(z)$.

$$\sum_{\alpha \in F^-} w'(\alpha) \geqslant -c_2(z).$$

All other faces and vertices have non-negative charge (Claim 4 and Lemma 2) and from $\sum_{\alpha \in F} w'(\alpha) + \sum_{x \in V} w'(v) = -12$ (Claim 2) it follows that

$$\begin{aligned}\sum_{\alpha \in F \setminus F^-} w'(\alpha) + \sum_{x \in V} w'(v) &= -12 - \sum_{\alpha \in F^-} w'(\alpha)\\ &\leqslant -12 + c_2(z).\end{aligned}$$

Note that for $k \geqslant 6$ every face $\alpha \in F_k$ has a charge of at least $k - 6$ (Claim 4). Furthermore, let v_{big} be the number of vertices with $d(x) \geqslant 42 + 21z$ where each of these vertices has a charge of at least $(2/7)z$ (Claim 3). Thus,

$$\begin{aligned}\sum_{k \geqslant 7} (k-6)|F_k| + (2/7)z\, v_{\text{big}} &\leqslant \sum_{\alpha \in F \setminus F^-} w'(\alpha) + \sum_{x \in V} w'(v)\\ &\leqslant -12 + c_2(z).\end{aligned}$$

Consequently, there is a k_z such that $F_k = \emptyset$ for every $k \geqslant k_z$. Furthermore, the number of faces $\alpha \in F_k$ for $k < k_z$ incident with vertices of degree at least $42+21z$ is bounded because the number of such vertices is bounded by the above inequality. In fact, the inequality gives even a bound for $|F_k|$ for all $k \geqslant 7$. Finally, the number of faces $\alpha \in F_k$ $(k < 7)$ incident only with vertices of 'small' degree $(< 42 + 21z)$ is bounded because every face-type is allowed to occur at most z times.

Altogether it turns out that for fixed z every z-oblique graph G contains a bounded number of faces and no face with $d(\alpha) \geqslant k_z$. However, the number of pairwise non-isomorphic graphs satisfying such a condition is obviously finite. This completes the proof of Theorem 1.

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